Original Research Paper

An algorithm shows prime number patterning

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Abstract: A sieve algorithm is created and implemented in Java. The stringent logic of the algorithm is the proof. From the sieves a sequence of families of linear equations is derived. These equations are envelopes to the patterning of prime numbers. Results of the algorithm are discussed.

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Introduction

Prime numbers can only be devided by one and itself, this excludes the number one. Large prime numbers are important in crypthography. Many properties of the patterning of prime numbers have been described but anything as simple as the linear equations of this paper could nowhere be found (Mahato and Shah, 2023).

Tools

Walksets

For the purpose of this sieve algorithm it was necessary to define sorted sets, which were coined "walksets" (W) as a distinction to sets which are always unsorted. Walksets can be thought of as a walk on the number ray and are different from intervals as they only can contain whole numbers or symbols. Walksets have direction, they can be emtpy, they are written with angle brackets. Infinity is only possible either at the start or end of a walkset or otherwise as a periodic term that encompasses the whole walkset.

$$W_{empty} := \langle\rangle \quad (1)$$
$$W_{natural\ numbers} := \langle 1, 2, \dots \infty\rangle \quad (2)$$
$$W_{symbol\ pattern\ 1} := \langle L, \dots \infty\rangle = \langle \overline{L}\rangle \quad (3)$$
$$W_{symbol\ pattern\ 2} := \langle \overline{LM}\rangle \quad (4)$$
$$W_{symbol\ pattern\ 3} := \langle \overline{MLMLMM}\rangle \quad (5)$$
$$W_{symbol\ pattern\ 4} := \langle \overline{LMLMMM}\rangle \quad (6)$$

Symbols

The symbol L means undetermined wether prime or composite number. The symbol M means composite number and the symbol P means prime number. The symbol 1 means the number 1 which is not a prime number.

Primorial

The size of the sieves are calculated by multiplying all prime numbers up to the lower bound of the sieve, thus the primorial.

Algorithm

The algorithm is named Heeren algorithm (HA) and a second implementation of the HA is called BLOX.

HA

The HA is shown in Fig. 1. These are the first steps:

$A_{start} = \langle\rangle$	$B_{start} = \langle\rangle$	$C_{start} = \langle 1, 2, \dots \infty\rangle$
$A_1 = \langle\rangle$	$B_1 = \langle 1\rangle$	$C_1 = \langle 2, 3, \dots \infty\rangle$
$A_2 = \langle 1\rangle$	$B_2 = \langle 2\rangle$	$C_2 = \langle 3, 4, \dots \infty\rangle$
$A_3 = \langle 1, 2\rangle$	$B_3 = \langle 3\rangle$	$C_3 = \langle 4, 5, \dots \infty\rangle$
$A_4 = \langle 1, 2, 3\rangle$	$B_4 = \langle 4\rangle$	$C_4 = \langle 5, 6, \dots \infty\rangle$
$A_5 = \langle 1, 2, 3, 4\rangle$	$B_5 = \langle 5\rangle$	$C_5 = \langle 6, 7, \dots \infty\rangle$

At the same time, the HA pattern walksets with symbols are kept synchronized with the numbers.

$AP_{start} = \langle\rangle$	$BP_{start} = \langle\rangle$	$CP_{start} = \langle \overline{L}\rangle$
$AP_1 = \langle\rangle$	$BP_1 = \langle 1\rangle$	$CP_1 = \langle \overline{L}\rangle$

AP_2 = <1> BP_2 = <P> CP_2 = <¯LM>
AP_3 = <1, P> BP_3 = <P> CP_3 = <¯MLMLMM>
AP_4 = < 1, P, P > BP_4 = <M>
CP_4 = <¯LMLMMM>
AP_5 = <1, P, P, M> BP_5 = <P>
CP_5 = <¯MLMMMLMLMMMLMLMMMLMMMMML MLMMMM>

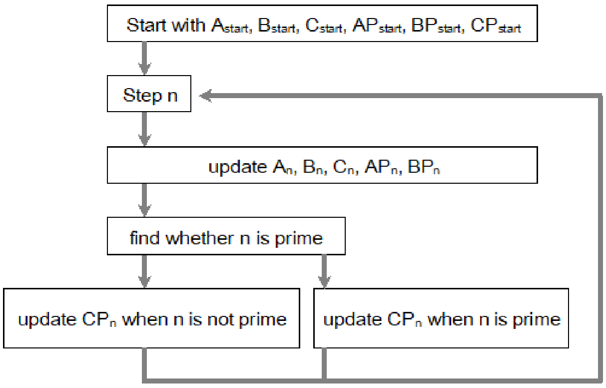

Fig. 1. The Heeren algorithm HA

Update of walkset A_{n-1} to A_n

The only element of B_{n-1} is cut from B_{n-1} and pasted into A_{n-1} on the right side. Thus set A_{n-1} becomes A_n . At the start B_{start} = <>. Since there is no element in B_{start} to cut A_1 is an empty walkset.

Update of walkset B_{n-1} to B_n

The leftmost element of C_{n-1} is cut and pasted into B_{n-1} left empty be the update of A. Thus B_{n-1} again contains one element that is the step number and thus becomes B_n . At the start B_{start} = <> therefore as B is filled for the first time it becomes B_1.

Update of walkset C_{n-1} to C_n

This has already taken place during the update of B by cutting the leftmost element of C_{n-1}. Thus C_{n-1} has become C_n.

Update of pattern walkset AP_{n-1} to AP_n

The only element of BP_{n-1} is cut from BP_{n-1} and pasted into AP_{n-1}. The new element has to become the rightmost element to keep AP in match with A. Thus AP_{n-1} becomes AP_n. At the start BP_{start} = <> therefore AP_1 is an empty set, since there is no element in BP_{start} to cut.

Update of pattern walkset BP_{n-1} to BP_n

The leftmost element of CPn-1 is copied (not cut ! see lemma) from CPn-1 and pasted into BP_{n-1} left empty by the update of AP. Thus BP_{n-1} again contains one element that is the type of the step number and thus becomes BP_n. At the start BP_{start} = <>, therefore as BP_{n-1} is filled for the first time it becomes BP_1.

Step number n = 3: <¯MLMLMM>
This means: <MLMLMM,MLMLMM, MLMLMM, M... ∞>
Removing the first letter leaves:
<LMLMM, MLMLMM, MLMLMM, M... ∞>
This can be rewritten as:
<LMLMMM, LMLMMM, LMLMMM, ... ∞>
Thus step number n = 4 is: <¯LMLMMM>

Lemma: The equivalence of cutting in C and copy-pasting in CP.

Find whether n is prime

The element n, that is the step number is contained in B_n. Its type information is contained in BP_n. If BP_n contains the type element M then the current step number is not prime. If BP_n contains the type element L then the current step number is prime. This is denoted by changing L into P in BP_n (with exception of number one which turns L into symbol 1).

Update pattern walkset CP_{n-1} to CP_n when n is not prime

<u>Procedure: move</u>
As can be seen in the description "Update of pattern walkset BP_{n-1} to BP_n" the type information of the step number n is not cut from CP_{n-1}. Instead this leftmost type information is moved to the rightmost place of the periodic term. Thus CP_{n-1} becomes CP_n (see lemma).

Update pattern walkset CP_{n-1} to CP_n when n is prime

<u>Procedure: move</u>
The first procedure is the same as for "n is not prime".
<u>Procedure: copy</u>
The sieve size is increased by copying the pattern and pasting it n-1 times to the right of itself.
<u>Procedure: change</u>
The types of all numbers x for which applies
$x \cdot n$ with ($x \in \mathbb{N} \wedge x > 1 \wedge x \cdot n \leq$ sieve $size_n + n$)
in CP_{n-1} are turned from undertermined L types into M types, unless they are already of type M. Thus CP_{n-1} becomes CP_n.

HA results

Let us think of the CPn patterns as sieves Fig. 2. The pattern of L is the envelope of all prime numbers above the step number n. The pattern of M is the area outside the prime number envelope.

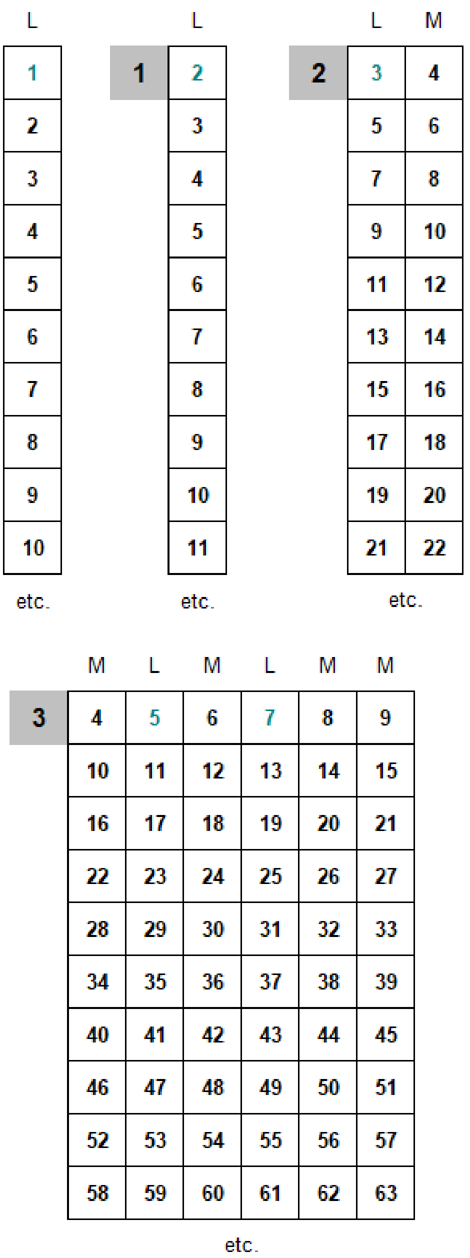

Fig. 2. The sieves of starting state, step n = 1 and step n = 2 and step n = 3.

At first sight the HA sieves could be mistaken for Eratosthenes sieves, but there are important differences. The HA has a lower bound, no upper bound and it consists of vertical columns, which can be described by linear equations. The L columns contain not only prime numbers but also composite numbers, therefore they are only envelopes to prime numbers.

Proof

The stringent logic of the algorithm is the proof.

Prime number envelopes

All envelope equations come directly from the HA sieves Fig. 2. At step number n = 1 the L column starts with 2 and then 3, 4, 5, to infinity. This leads to the trivial prime number envelope equation with $x \in \mathbb{N}_0$

$$f(x)=1x+2 \quad (7)$$

Fig. 2 at step number 2 the L column starts with 3, the width of the sieve is 2 this leads to the envelope equation with $x \in \mathbb{N}_0$

$$f(x)=2x+3 \quad (8)$$

Fig. 2 at step number 3 the L columns start with 5 and 7, the width of the sieve is 6 this leads to a family of prime number envelope equations with $x \in \mathbb{N}_0$.

$$f(x)=6x+5 \quad (9)$$

$$f(x)=6x+7 \quad (10)$$

Followed by an even larger family of equations for step number n = 5 envelope (no Fig.) with $x \in \mathbb{N}_0$.

“Eq. 9” spawns:

$$f(x)=30x+11 \quad (11)$$

$$f(x)=30x+17 \quad (12)$$

$$f(x)=30x+23 \quad (13)$$

$$f(x)=30x+29 \quad (14)$$

“Eq. 10” spawns:

$$f(x)=30x+7 \quad (15)$$

$$f(x)=30x+13 \quad (16)$$

$$f(x)=30x+19 \quad (17)$$

$$f(x)=30x+31 \quad (18)$$

The sieve size increases from step to step with n primorial. Also for each following step of HA the number of prime number envelope equations increases rapidly.

For each prime number the envelopes can be determined. For example prime number 193877777 belongs to envelopes with $x \in \mathbb{N}_0$.

$$f(x)=1x+2 \quad (19)$$

$$f(x)=2x+3 \quad (20)$$

$$f(x)=6x+5 \quad (21)$$

$$f(x)=30x+17 \quad (22)$$

$$f(x)=210x+107 \quad (23)$$

$$f(x)=2310x+1787 \quad (24)$$

Composites

The All M columns by the logic of the HA contain only composite numbers (gaps).

Hypotheses

The following equations are hypothesized with $x \in \mathbb{N}_0$ and are named latisses. With each step of HA the number of latisses increases rapidly.

$6x_1+7 = 6x_2+5 * 6x_3+5$ (25)

$6x_1+7 = 6x_2+7 * 6x_3+7$ (26)

$6x_1+5 = 6x_2+5 * 6x_3+7$ (27)

It is hypothesized that all semi prime numbers are contained in the L columns and thus within the prime number envelope.

Prime gaps, three consecutive primes and twin primes

We define letters a, b, c and d.

a := <LMLMMM> (28)

b := <LMMMMM> (29)

c := <MMLMMM> (30)

d := <MMMMMM> (31)

Because step number n ist larger than two with each step only one L of letter a can be turned to M, Fig. 3.

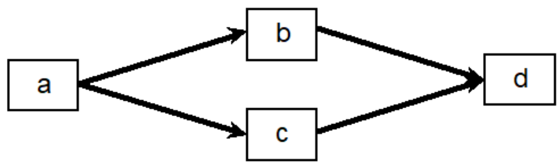

Fig. 3. Letter a can turn into b or c. Letters b and c can turn into letter d. Letter d can not change anymore.

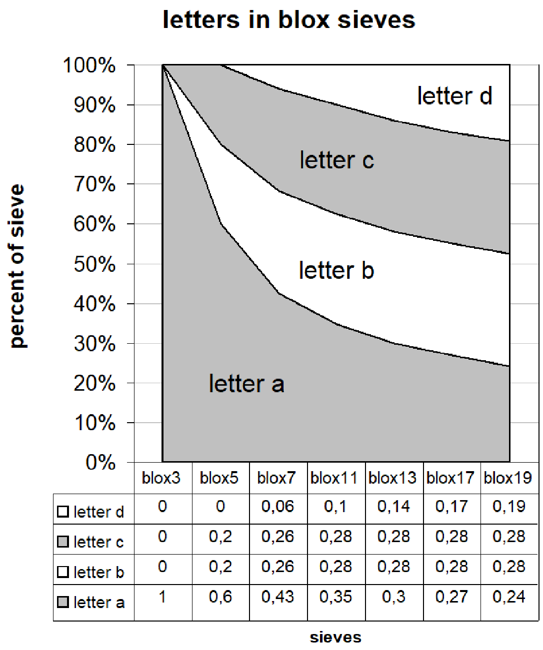

Fig. 4. How letter a decreases from sieve to sieve and letters b, c and d increase.

Letter d and clusters of letter d are prime gaps. As HA walks from step to step all prime gaps are multiplied and transferred to the next sieve. Thus any large prime gap that was discovered will appear again and again and can only become larger.

It holds true that no matter in which sequence the letters a, b, c, and d are arranged it is impossible to build the pattern of three consecutive primes <LMLML> beyond 3, 5, and 7.

For twin primes envelope Fig. 4. <LML> which appears only in letter a it can be said, that the density in the natural numbers decreases with each step of HA, as relatively more letters a turn into b, c and ultimately d. The behavior toward infinity (limit) is unkown to the author.

Factorization of semiprimes

Semiprimes are the product of two prime numbers. Semiprimes up to 12 digits with similar factor sizes were factorized by making use of latisses and targeted trial and error to determine the x_2 and x_3 values of the right hand side, which can be computed in parallel.

For example the semi prime 82557089 was found to be of latisses.

$6x_1+5 = 6x_2+5 \cdot 6x_3+7$ (32)

and

82557089 = (2310*4 + 839) * (2310*3 + 1261) (33)

leading to prime number factors

82557089 = (10079) * (8191) (34)

prime number production

For the purpose of using large sieves the BLOX variant of HA was developed. Each sieve is a file that contains only the L columns. The computation was started with a file at step n = 3 containing 5 and 7 and ended through successive computation at step n = 29 with 667 files, each containing approximately about 1.6 million L entries in sieve blox29.

The blox computation is done by the equivalence of HA. From n = 3 to n = 5 five copies of the L-columns are made in the following way. The sievesize of n = 3 that is 6 is used. Five and composites of five are cut.

Blox5 is:

5	cut
7	7
5+6=11	11
7+6=13	13
5+2*6=17	17
7+2*6=19	19
5+3*6=23	23
7+3*6=25	cut
5+4*6=29	29
7+4*6=31	31

For blox7 the sievesize of n = 5 is 30. Seven copies are made. 7 and composites of 7 are cut. And so on and so forth. The BLOX sieves contain all prime numbers in the range of the sieve and some composite numbers.

With blox29 three cases were considered.

1.) Picking 310 digit prime numbers at random from all natural numbers (0.12% success rate). Fig. 5.
2.) Picking 310 to 312 digit prime numbers at random from inside the envelope with sieve blox29 (0.9% success rate) Fig. 6.
3.) Picking 310 to 312 prime numbers outside the envelope with blox29, where no prime numbers in 200,000 picks could be found.

The picking of large, here 310 to 312 digit primes is done the following way. A large sieve size 743 primorial was used which has 310 digits (= sievesize). A number from the blox29 picked at random (= L) is used and a random integer factor between 0 and 99 (= x). The formula for the likelyprime is:

$$\text{likelyprime} = \text{sievesize}_{743} * x + L_{blox29} \qquad (32)$$

This formula is an envelope equation of very large size. Because sievesize_{743} contains the factor 29 primorial, the likelyprime is picked out of an L column of blox29, thus inside the envelope of blox29. By using different sievesizes and different x factors, all prime numbers possible can be picked.

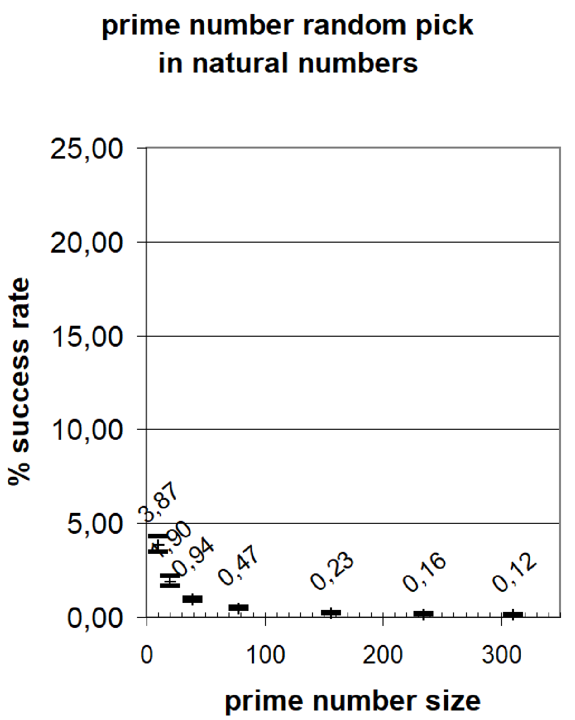

Fig. 5. Prime number random pick in natural numbers

for sieve sizes 10, 20, 39, 78, 156, 234, and 310 1000 picks each with standard deviation.

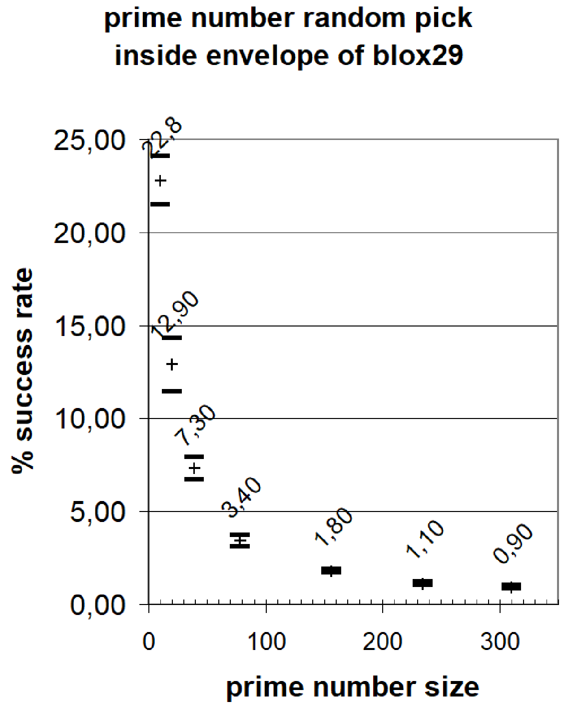

Fig. 6 Prime number random pick in envelope blox29 for sieve sizes 10, 20, 39, 78, 156, 234, and 310 1000 picks each with standard deviation

The holes in the envelopes

For envelopes equations (9) and (10) the holes follow a simple emergence law, which is an algorithm, Fig. 7 and Fig. 8.

In Fig. 7 it starts $f_{(x)}=6x+5$ with x = 0 $f_{(x)}=5$. Now every 5^{th} number is a composite. The first composite is 35 = 5*7. From this point on every 7^{th} number is a composite, too. The next composite in Fig. 7 is 65=5*13. From this point on every 13^{th} number is composite, too. The next composite is 77=7*11. From this point on every 11^{th} number is a composite, too. And so on and so forth.

In Fig. 8 it starts starts $f_{(x)}=6x+7$ with x = 0 $f_{(x)}=7$. Now every 7^{th} number is a composite. Keeping in logic with latisses (25) and (26) the first composite comes from 25=5*5. From this point on every 5^{th} number in the envelope is a composite. The next composite is 49=7*7, nothing new happens, because we already got the every 7^{th} pattern. But the next composite is 55=5*11. From this point on every 11^{th} number is a composite, too. And so on and so forth.

The emergence law of composites inside the envelopes very much remindes of the emergence law of composites in all natural numbers. But the order of prime numbers appearing is different. At Fig. 7 it is: 5, 7, 13, 11, 19, 17, 31, 23, 37, 29, 43, … . At Fig. 8 it is: 7, 5, 11, 17, 13, 23, 19, 29, 41, 31, 37,… .

Conclusion

Having derived not only the prime number envelopes from HA but also having presented the emergence law of its holes (= composite numbers), it can be said, the patterning of prime numbers is described in a basic way. Now the higher mathematics can be put into place.

This work approaches prime number patterning from the perspective of a software developer and scientist. It is presented to the world of higher mathematics passing the baton.

Funding Information

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References

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	5x	7x	13x	11x	19x	17x	31x	23x	37x	29x	43x
	5										
	11										
	17										
	23										
	29										
5*7	35	35									
	41	41									
	47	47									
	53	53									
	59	59									
5*13	65	65	65								
	71	71	71								
7*11	77	77	77	77							
	83	83	83	83							
	89	89	89	89							
5*19	95	95	95	95	95						
	101	101	101	101	101						
	107	107	107	107	107						
	113	113	113	113	113						
7*17	119	119	119	119	119	119					
5*5*5	125	125	125	125	125	125					
	131	131	131	131	131	131					
	137	137	137	137	137	137					
11*13	143	143	143	143	143	143					
	149	149	149	149	149	149					
5*31	155	155	155	155	155	155	155				
7*23	161	161	161	161	161	161	161	161			
	167	167	167	167	167	167	167	167			
	173	173	173	173	173	173	173	173			
	179	179	179	179	179	179	179	179			
5*37	185	185	185	185	185	185	185	185	185		
	191	191	191	191	191	191	191	191	191		
	197	197	197	197	197	197	197	197	197		
7*29	203	203	203	203	203	203	203	203	203	203	
11*19	209	209	209	209	209	209	209	209	209	209	
5*43	215	215	215	215	215	215	215	215	215	215	215
13*17	221	221	221	221	221	221	221	221	221	221	221
	227	227	227	227	227	227	227	227	227	227	227
	233	233	233	233	233	233	233	233	233	233	233
	239	239	239	239	239	239	239	239	239	239	239
5*7*7	245	245	245	245	245	245	245	245	245	245	245
	251	251	251	251	251	251	251	251	251	251	251
	257	257	257	257	257	257	257	257	257	257	257
	263	263	263	263	263	263	263	263	263	263	263
	269	269	269	269	269	269	269	269	269	269	269
5*5*11	275	275	275	275	275	275	275	275	275	275	275
	281	281	281	281	281	281	281	281	281	281	281

Fig. 7 The holes in envelope 6x+5 up to 281

	7x	5x	11x	17x	13x	23x	19x	29x	41x	31x	37x
	7										
	13										
	19										
5*5	25	25									
	31	31									
	37	37									
	43	43									
7*7	49	49									
5*11	55	55	55								
	61	61	61								
	67	67	67								
	73	73	73								
	79	79	79								
5*17	85	85	85	85							
7*13	91	91	91	91	91						
	97	97	97	97	97						
	103	103	103	103	103						
	109	109	109	109	109						
5*23	115	115	115	115	115	115					
11*11	121	121	121	121	121	121					
	127	127	127	127	127	127					
7*19	133	133	133	133	133	133	133				
	139	139	139	139	139	139	139				
5*29	145	145	145	145	145	145	145	145			
	151	151	151	151	151	151	151	151			
	157	157	157	157	157	157	157	157			
	163	163	163	163	163	163	163	163			
13*13	169	169	169	169	169	169	169	169			
5*5*7	175	175	175	175	175	175	175	175			
	181	181	181	181	181	181	181	181			
11*17	187	187	187	187	187	187	187	187			
	193	193	193	193	193	193	193	193			
	199	199	199	199	199	199	199	199			
5*41	205	205	205	205	205	205	205	205	205		
	211	211	211	211	211	211	211	211	211		
7*31	217	217	217	217	217	217	217	217	217	217	
	223	223	223	223	223	223	223	223	223	223	
	229	229	229	229	229	229	229	229	229	229	
	235	235	235	235	235	235	235	235	235	235	
	241	241	241	241	241	241	241	241	241	241	
13*19	247	247	247	247	247	247	247	247	247	247	
11*23	253	253	253	253	253	253	253	253	253	253	
5*37	259	259	259	259	259	259	259	259	259	259	259
	265	265	265	265	265	265	265	265	265	265	265
	271	271	271	271	271	271	271	271	271	271	271
	277	277	277	277	277	277	277	277	277	277	277
	283	283	283	283	283	283	283	283	283	283	283

Fig. 8 The holes in envelope 6x+7 up to 283